

# Representation and design of heterogeneous components

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## Abstract

Recent advances in rapid prototyping techniques enable the fabrication of components whose composition can be controlled and varied in any desired fashion. This presents design engineers with the opportunity and the challenge to design heterogeneous components that optimize some design objective or can provide certain functionality. However, there are no well-established techniques for representing heterogeneous components or design tools to arrive at the appropriate composition distribution. A shape and composition model based on iso-parametric interpolation functions is presented that can be used in conjunction with non-linear programming algorithms to automatically compute optimal composition distribution.

## 1. Introduction

Limitation of manufacturing processes is one of the reasons why most traditional structural and mechanical components are made of homogeneous materials. Even though in many applications appropriately designed heterogeneous components may be more optimal from a functional point of view, manufacturing considerations have traditionally rendered such designs infeasible. However, with the advent of rapid prototyping or solid freeform fabrication (SFF) processes (Kochan 1993, Ashley 1991) many of the traditional manufacturing constraints are no longer applicable. SFF processes build parts by adding material to the part layer-by-layer. One of the promises of such layered manufacturing methods is that it will be possible to fabricate heterogeneous components or functionally gradient materials for which the material composition within the component can be varied in a desired fashion. While most of the SFF techniques are theoretically capable of composition control within manufactured parts, powder based processes and fused deposition processes appear to be particularly promising. Progress towards composition control of heterogeneous components have been reported recently using SFF processes such as Selective Laser Sintering (SLS) (Jepson et al, 1997), 3D printing (Yoo et al, 1998), Shape Deposition Manufacturing (SDM) (Fessler et al 1997), Direct Light Fabrication (Lewis and Nemme, 1997) and Laser Engineered Net Shaping (LENS) (Griffith et al, 1997) etc. Other specialized processes such as Fused Deposition of Ceramics (Gasdaska et al, 1998) and Direct Metal Deposition (Mazumder et al, 1997) have also been applied to manufacture multi-material components.

Material composition distribution can be thought of as a volumetric property of the solid since it varies within the part and has to be expressed as a function of the spatial coordinates. Most traditional designs do not have volumetric properties and therefore the

need for expressing such properties did not exist. Traditional geometric modeling systems and design tools therefore do not support the ability to model, analyze or design heterogeneous components. Recently, methods for modeling such components have been explored. Kumar V. and Dutta D. (1998) have presented an extension of the traditional solid model representation based on regular sets (r-sets), to define  $r_m$ -sets that include material data. They have defined set operations for combining these  $r_m$ -sets to define heterogeneous solids. Jackson et al (1998) present an alternate representation where they subdivide a solid model into simpler domains over which composition functions are defined using Bernstein polynomials as basis functions.

In this paper, a representation of material distribution is presented that is based on a finite element mesh generated over the solid or a feasible region. The material composition data at the nodes are interpolated to obtain composition data within each element. The nodal values of composition data can be modified to vary the composition distribution pattern within the solid. In addition, one can also perform finite element analysis using the mesh to determine structural, thermal or dynamic properties of the resultant solid. Finally, by defining the design as an optimization process where material composition at the nodes are treated as the design variables, it is possible to compute the optimal composition distribution that maximize or minimizes the design objective while satisfying the design constraints. Kumar and Gossard (1996) have used a similar representation to model geometry using shape density function.

In section 2, the shape and composition representation is described. Analysis of heterogeneous components using finite element method is described in section 3. A method for designing these components by optimizing the composition distribution is considered in section 4. A few examples are presented in section 5, where the design of a porous component is considered. Finally, conclusions are presented in section 6.

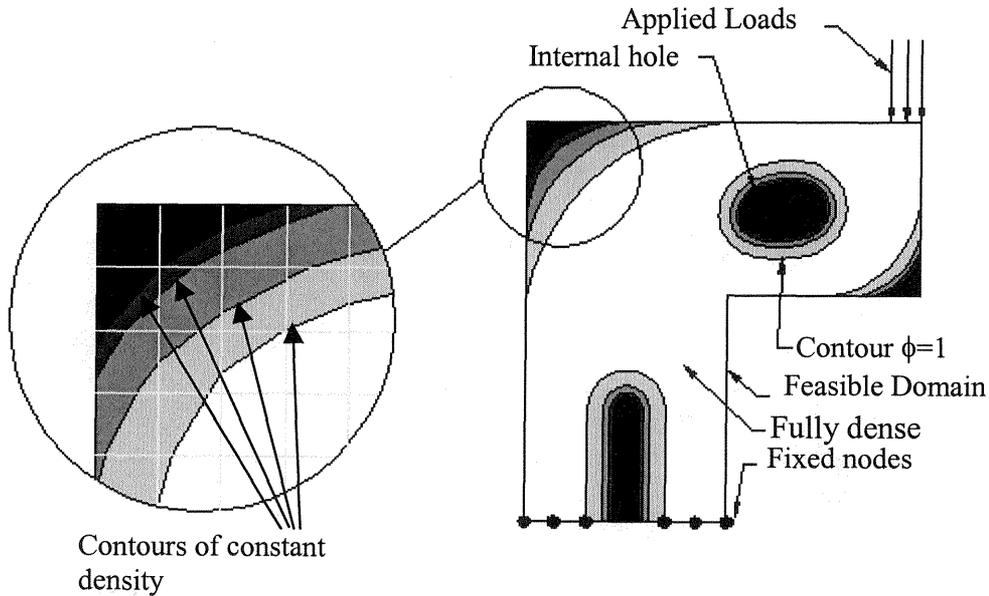
## **2. Shape and composition representation**

Volumetric properties can be expressed as a scalar or vector field defined over the volume of the solid. Therefore, varying density may be defined as a scalar field  $\phi = \phi(x, y, z)$ , while the directions of anisotropy, fiber orientation of a composite material or material composition of multi-material component may be express as a vector field  $\mathbf{v} = \mathbf{v}(x, y, z)$ . In order to define such fields, a piece-wise interpolation scheme could be used similar to that used in finite element method. The volume is subdivided into simpler cells (or elements) and the field within each cell is interpolated based on the values at the nodes (see Bathe, 1996 or Carey and Oden, 1981). This requires augmenting boundary representation of solids with a finite element like mesh as well as data on nodal values of the volumetric properties.

In the general case, when the component has 'n' materials, a vector of length n can express the composition at any point. The first n-1 members of the vector represent the volume fraction of the first n-1 materials that make up the composite and the nth member represents the void fraction. For example, if the material is porous so that it is made of one material and void, its composition can be expressed as either the void fraction or the density of the material so that the composition distribution, in this case, can be expressed

using a scalar field. We will illustrate the composition representation, analysis and optimization using this example, where the composition is expressed as a density field  $\phi(\mathbf{x})$ . Extension to the general case is possible using the methods described here.

Figure 1 illustrates the geometry and composition representation. The inverted L-shaped region shown is the feasible domain within which the geometry is defined. The figure shows contours of the density function corresponding to constant values of densities. The fully dense regions (where  $\phi=1$ ) are shown in white color while the black regions have density equal to zero and are therefore void of material. All other density values are represented using a grayscale color distribution with lighter shades representing higher density than darker shades.



**Figure 1: Shape Representation**

In order to define the density function within the feasible domain, the domain is divided into a quadrilateral mesh as illustrated in the magnified section of Figure 1 and the nodal values of density are specified. The density function distribution within each element is obtained by interpolating the nodal values. Using four-node quadrilateral elements, the density function within each element can be expressed as,

$$\phi(s, t) = \phi_1 N_1(s, t) + \phi_2 N_2(s, t) + \phi_3 N_3(s, t) + \phi_4 N_4(s, t) \quad (2.1)$$

where,  $\phi_i$  are the nodal density values and  $N_i(s, t)$  are the isoparametric shape functions for the four-node quadrilateral element expressed in terms of the parametric coordinates  $s$  and  $t$  (Bathe, 1996). The shape functions can be expressed as

$$\begin{aligned}
N_1 &= \frac{1}{4}(1+s)(1+t), & N_2 &= \frac{1}{4}(1-s)(1+t); \\
N_3 &= \frac{1}{4}(1-s)(1-t), & N_4 &= \frac{1}{4}(1+s)(1-t)
\end{aligned}
\tag{2.2}$$

For an isoparametric element, the mapping between the parametric space and the real coordinates  $(x, y)$  is defined by

$$x = \sum_{i=1}^4 x_i N_i(s, t) \quad \text{and} \quad y = \sum_{i=1}^4 y_i N_i(s, t)
\tag{2.3}$$

The design problem definition consists of specifying the feasible domain, the loads the structure has to carry and the displacement boundary conditions that describe how the structure is supported. In Figure 1, the arrows at the top represent the load supported by the structure. The solid circles represent nodes that are constrained to have zero displacement.

### 3. Analysis using Finite Element Method

The finite element mesh used to represent the shape and the composition distribution can also be used to perform a finite element analysis to obtain structural and other mechanical properties of the solid. In doing so, firstly, the variation of material properties as a function of composition must be established either experimentally or using appropriate analytical models. In this paper, we assume polynomial material property versus density relation. The real relation depends on the microstructure and may vary if the component is made using different techniques even if the composition is identical.

In constructing the stiffness matrix, we integrate the principle of virtual work, over each element to obtain the element stiffness matrix and then assemble these together to obtain the global stiffness matrix. We have used four-node iso-parametric quadrilateral elements for the analysis. Therefore, the displacements are represented using the same piece-wise bilinear interpolation that was used to represent the composition distribution (equation 2.2) within each element. This representation yields the following strain-displacement relation for the element,

$$\{\varepsilon^e\} = [\mathbf{B}] \{u_h^e\}
\tag{3.1}$$

where,  $\{u_h^e\}$  is the displacement vector corresponding to the quadrilateral finite element. The governing equations can be expressed in general using the following type of weak form of a variational principal.

$$\int_V \{\delta \mathbf{X}\}^T [\mathbf{B}]^T [\mathbf{D}(\phi)] [\mathbf{B}] \{\mathbf{X}\} dV = \int_S \{\delta \mathbf{X}\}^T \{\mathbf{f}\} dS + \int_V \{\delta \mathbf{X}\}^T \{\mathbf{b}\} dV
\tag{3.2}$$

The above equation represents the weak form applied to an element.  $V$  is the volume of the element,  $S$  is the surface area of the element that lies on the boundary of the object,  $\{\mathbf{f}\}$  is the traction acting on these boundaries and  $\{\mathbf{b}\}$  is the body force acting on the

element. The vector  $\{\mathbf{X}\}$  contains the nodal variables while the  $\{\delta\mathbf{X}\}$  vector contains the corresponding virtual variables. The  $[\mathbf{D}(\phi)]$  matrix contains material properties associated with the analysis.

The matrix  $[\mathbf{D}]$  depends on the material properties and therefore is a function material composition (density in our example). The left-hand side of equation (3.2) needs to be integrated to determine the stiffness matrix. The integration was performed using Gauss quadrature algorithm (Bathe, 1996). In order to determine the correct order of quadrature to use, the degree of the polynomial terms of the  $[\mathbf{B}]^T[\mathbf{D}(\phi)][\mathbf{B}]$  matrix must be determined. Since density is a linear function in the parametric coordinates, the terms in the matrix  $[\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]$  are polynomials of degree  $2+n$  where  $n$  is the degree of the polynomial assumed in the material property-composition relation. Gauss quadrature of order  $m$  (using  $m \times m$  integration points) can integrate a polynomial of degree  $2m-1$  accurately. Therefore, if we choose polynomial material property-density relation of degree 4, the terms in  $[\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]$  matrix are polynomial of degree 6, which can be integrated using fourth-order Gauss quadrature or 16 integration points.

#### 4. Optimization of composition distribution

Finite element analysis described in the previous section can yield information about the mechanical properties of heterogeneous components. However, from a design perspective one would like to compute the composition distribution that would induce the desired property. For example, one may solve for the optimal density distribution with the objective of minimizing the compliance of the structure subject to a limit on its volume or mass. Minimizing the compliance is the same as maximizing the stiffness of the structure. This is a very common objective function used in structural optimization and has been used extensively in topology optimization research (Bendsoe and Kikuchi, 1988, Kumar and Gossard, 1996). Below we illustrate the process by minimizing compliance of a planar structure. The minimization of compliance may be written as,

$$\text{Minimize } L(\mathbf{u}(\phi)) = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}(\phi) d\Omega + \int_{\Gamma} \mathbf{t} \cdot \mathbf{u}(\phi) d\Gamma \quad (4.1)$$

subject to,

$$M(\phi) = \int_{\Omega} \phi d\Omega \leq M_0, \quad M_0 = (1 - \alpha)M_i, \quad \text{where } M_i = \int_{\Omega} d\Omega \quad (4.2)$$

$$\int_{\Omega_0} \{\delta\epsilon\}^T [\mathbf{D}(\phi)] \{\epsilon\} d\Omega_0 = L(\delta\mathbf{u}) \quad (4.3)$$

$$0 \leq \phi \leq 1 \quad (4.4)$$

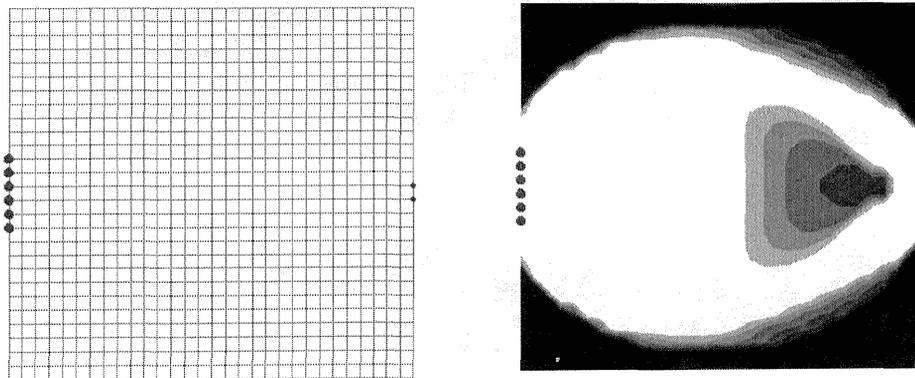
$L(\mathbf{u})$ , the mean compliance, is twice the work done by the applied forces (traction  $\mathbf{t}$  and body force  $\mathbf{f}$ ) during the displacement  $\mathbf{u}$ . Equation (4.2) describes the constraint that the mass  $M$  of the optimal geometry should be less than or equal to  $M_0$ . In order to remove  $\alpha$  percentage of mass from the initial mass  $M_i$ , we set  $M_0$  to  $(1-\alpha)M_i$ .  $\{\epsilon\}$  and  $\{\delta\epsilon\}$  are the strain and virtual strain in the structure caused by the displacement  $\mathbf{u}$  and the virtual displacement  $\delta\mathbf{u}$  respectively.  $[\mathbf{D}(\phi)]$  is the matrix of elasticity constants that relate

stresses and strains for a linear elastic material. We assume that these elasticity constants are functions of the density.

The optimization problem given by the equations (4.1) through (4.4) can be solved using a nonlinear programming algorithm. Each evaluation of the objective function requires a computationally expensive finite element analysis. For this reason, an optimization algorithm that does not require excessive objective function evaluations is desired. Our work uses a variation of sequential linear programming, which is described in Kumar (1993).

## 5. Examples and Results

Examples of optimizing the density distribution with the objective of minimizing the compliance are presented here. Figure 2 shows a rectangular geometry that is loaded like a cantilever beam so that it can be modeled as a plane stress problem. The rectangle is 10x9 units in dimension and is divided into 810 elements. The nodes on the left marked using solid circles were constrained to have no displacements and two nodes on the right were subjected to loads acting vertically downwards of magnitude 1000 units (shown using solid circles with lines drawn in the direction of the force). The Young's modulus of the fully dense material was assumed to be  $3 \times 10^7$  units and Poisson's ratio of 0.3. The Young's modulus was assumed to vary linearly with the relative density  $\phi$ . The optimal density distribution is shown in Figure 2 when computed using the constraint that the final mass of the structure should only be 70% of the mass of the fully dense structure.



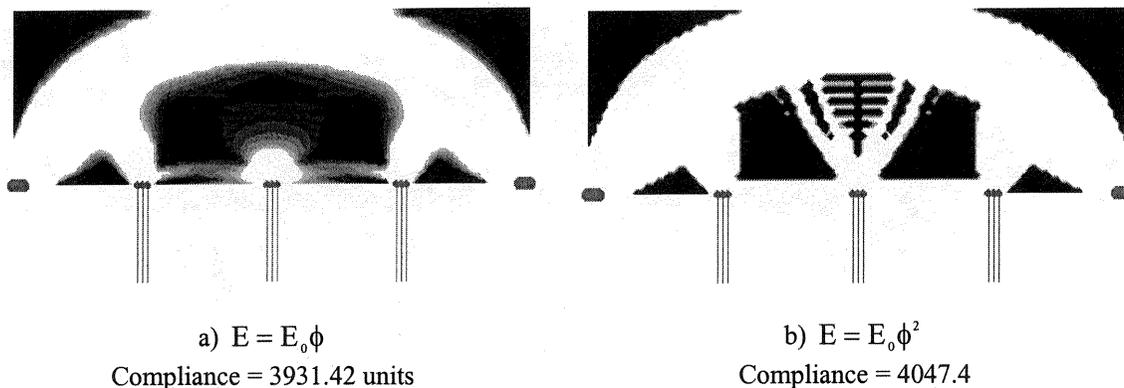
a) Mesh, Load and Boundary conditions

b) Optimal density distribution

**Figure 2: Optimal density distribution for a cantilever**

In figure 3, the design of a beam-like structure subjected to three sets of transverse loads is considered. A rectangular geometry, whose dimensions were 300 x 100 units, was divided into a uniform quadrilateral mesh containing 3333 nodes and 3200 elements. Three nodes on the lower two corners were constrained to have zero displacement (shown as solid circles in the figure). Three sets of forces were applied to the lower edge, each set consisting of three point loads (100 units each) applied over three nodes. The

structure was modeled as a plane stress problem. The optimal density distribution that minimizes the compliance was computed subject to the constraint that the final mass of the structure should be only 30% of the mass of the fully dense structure. The Young's modulus,  $E$ , of the fully dense material was assumed to be 10,000 units. When it was assumed that Young's modulus varied linearly with relative density  $\phi$ , the result shown in figure 3 (a) was obtained. If the region that is almost black is considered void of material (since the density there is very low) we obtain a structure within which the density is varying gradually from zero to one. However, when a quadratic relation was assumed, we obtained a fully dense (almost homogeneous) structure, as shown in figure 3(b).



**Figure 3: Optimal density distribution for a frame like structure**

It is clear that the material property-density relation has a significant influence on the optimal geometry computed. Notice that the heterogeneous design obtained by assuming a linear  $E$  vs.  $\phi$  relation has a lower compliance (is more stiff) than the fully dense homogeneous design obtained using a quadratic relation.

## 6. Conclusion

In this paper, some of the advantages of representing shape and composition distribution using a finite element mesh and its associated interpolation functions are described. Even though only four-node elements were used in this paper, it is possible to use higher order two and three-dimensional elements and their interpolations functions to model arbitrary heterogeneous solids. A material with varying density or void fraction was considered as an example here but future work will focus on using this representation to model multi-material composites. Design of heterogeneous components for other structural and non-structural applications and the associated design objectives and constraints also needs to be studied in the future.

## 7. Acknowledgements

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